THERMAL STRESSES IN AN ELASTIC HALF-SPACE

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The method of thermal fracture of brittle bodies becomes more and more widespread in recent times (see [1], for instance). Here the analysis of stress fields and of domain fracture occupies an important place. And since the fracture time is ordinarily quite small, then obtaining simple asymptotic expressions for the stresses in small times becomes quite valuable. Even more so when a real analysis by formulas expressing the exact solution is made extremely difficult, as a rule, by the necessity to integrate over infinite domains and to perform summations for finite awkward expressions containing singularities.

The model of an elastic half-space with local heating of the surface [1] is quite widespread in investigations of thermal fracture from the macroscopic aspects of strength criteria.

The first term of asymptotic expansions of the temperature and stress is found in [2] for boundary conditions of heat conduction of the first kind. The examination of conditions of the third kind is also of interest [1].

1. We consider the elastic half-space $z \geqslant 0$ on whose boundary heat transfer occurs according to Newton's law from the medium $z < 0$ in the cylindrical r, φ, z coordinates. The temperature of the medium is $\Theta = \Theta_0 f(r)$, where the domain of values of $f(r)$ is the segment [0, 1]. Find the temperature and stress within the elastic half-space whose initial temperature is $T = 0$.

This problem is solved in [3] for

$$f(r) = \exp(-r^2/4\delta) \quad (\delta = \text{const}). \tag{1.1}$$

The exact solution for small times is then represented in the form of convergent series. Replacing the series by finite sums, the author obtained an approximate solution whose error was not determined. In said work [3], exact and approximate solutions were postulated for certain classes of functions $f(r)$. For this case, an investigation into the accuracy of the approximation is in progress.

2. It is convenient to reduce the heat-conduction boundary-value problem

$$\begin{gathered}\partial T/\partial t = a\Delta T, \quad T|_{t=0} = 0,\\ \partial T/\partial z|_{z=0} = h(T|_{z=0} - \Theta), \quad T_{z=\infty} = T_{r=\infty} = 0\end{gathered} \tag{2.1}$$

to dimensionless quantities by setting

$$\Theta' = \Theta/\Theta_0,\ T' = T/\Theta_0,\ z' = z/\sqrt{\delta},\ h' = h\sqrt{\delta},\ t' = at/\delta, \tag{2.2}$$

where $\sqrt{\delta}$ is a certain characteristic dimension as in (1.1), say. Then, omitting the primes for brevity in (2.2), we write (2.1) in the form

$$\begin{gathered}\partial T/\partial t = \Delta T, \quad T|_{t=0} = 0,\\ \partial T/\partial z = h(T|_{z=0} - f(r)),\ T_{z=\infty} = T_{r=\infty} = 0.\end{gathered} \tag{2.3}$$

Applying the Laplace transform in t to (2.3), we obtain

$$* = \Delta T^*,\ \partial T^*/\partial z = h(T^*|_{z=0} - f(r)/s)\ (T^* = L_s[T]), \tag{2.4}$$

where L_s is the operator of the Laplace transform with parameter s.

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In turn, we apply the zero order Hankel transform in r to (2.4). We have

$$d^2\widetilde{T}^*/dz^2 = (s + \lambda^2)\widetilde{T}^*, \ d\widetilde{T}^*/dz|_{z=0} = h(\widetilde{T}^*|_{z=0} - \widetilde{f}(\lambda)/s), \tag{2.5}$$

where $\widetilde{T}^* = H_\lambda[T^*]$; H_λ is the Hankel transform operator with parameter λ:

$$\widetilde{f}(\lambda) = H_\lambda[f(r)] = \int_0^\infty rf(r)J_0(\lambda r)\,dr. \tag{2.6}$$

Solving the ordinary differential equation (2.5) and inverting the Hankel transform, we find

$$T^* = \int_0^\infty \lambda\widetilde{f}(\lambda)J_0(\lambda r)\frac{he^{-z\sqrt{s+\lambda^2}}}{s(\sqrt{s+\lambda^2}+h)}\,d\lambda. \tag{2.7}$$

Substituting (1.1) for $\delta = 1$, which corresponds to dimensionless coordinates, into (2.6) we obtain [4]

$$\widetilde{f}(\lambda) = 2\exp(-\lambda^2). \tag{2.8}$$

Then (2.7) agrees with the expression from [3]. This circumstance should be used to determine the original of the temperature, and the Laplace transforms and originals in the general case. They are obtained from those found in [3] by the formal replacement of (2.8). These formulas are not presented here because of their moderate practical value since they require the evaluation of improper integrals of awkward expressions containing eliminable singularities.

Let us note that it is sufficient for the existence of the direct and inverse Hankel transform that f(r) be a function of bounded variation and integrable on the half-line $[0, \infty)$ [5].

Another method of obtaining the exact solution is described in [2]. However, it also reduces to relationships possessing the disadvantages described above, in full measure. Hence, an asymptotic for $t \to 0$ is shown in [6] for the case of giving the temperature on the half-space boundary.

Convergence of the integrals in whose terms the exact solution is expressed is to be verified. Such a verification is performed in all the examples presented here.

3. Henceforth, we limit ourselves to functions f(r) for which the integrals

$$A_{ij}(r) = \int_0^\infty \lambda^i\widetilde{f}(\lambda)J_j(\lambda r)\,d\lambda, \quad B_{ij}(r, z) = \int_0^\infty \lambda^i\widetilde{f}(\lambda)J_j(\lambda r)\,e^{-\lambda z}d\lambda$$

converge for $i = 1, 2, \ldots, j = 0, 1$. For these functions an approximate solution is obtained successfully for small t which is more convenient for computations than is the exact solution.

Let L_t^{-1} denote the inverse operator to L_s. Here t is the argument of the original. We then have from (2.7)

$$T = \int_0^\infty \lambda\widetilde{f}(\lambda)J_0(\lambda r)\,d\lambda\int_0^t e^{-\lambda^2\tau}L_\tau^{-1}\left[\frac{he^{-z\sqrt{s}}}{\sqrt{s}+h}\right]d\tau,$$

from which, by setting

$$f_m(z, t) = L_t^{-1}\left[\frac{he^{-z\sqrt{s}}}{(\sqrt{s}+h)s^m}\right] \quad (m = 0, 1, \ldots),$$

we find

$$T = \int_0^\infty \lambda \tilde{f}(\lambda) J_0(\lambda r)\,d\lambda \int_0^t \left[\sum_{n=0}^{N} (-1)^n \frac{\lambda^{2n}\tau^n}{n!} f_0(z,t) + (-1)^{N+1} \frac{\lambda^{2N+3}\tau^{N+1}}{(N+1)!} f_0(z,t)\,e^{-\lambda^2\xi} \right] d\tau \quad (0<\xi<\tau) \tag{3.1}$$

by the Taylor theorem.

By using the obvious equality

$$\int_0^t f_k(z,\tau)\,d\tau = f_{k+1}(z,t) \quad (k=0,1,\ldots),$$

we can prove by induction that

$$\varphi_n(z,t) = (-1)^n \int_0^t f_0(z,\tau)\,\tau^n d\tau = \sum_{k=0}^{n} \frac{(-1)^{k+n} n!}{(n-k)!} f_{k+1}(z,t) \qquad (n=0,1,\ldots),$$

from which and from the boundedness of $f_0(z, \tau)$ (see [7]) there follows

$$\varphi_n = o(t^{n+1}) \quad (t \to 0).$$

Integrating (3.1) term by term, we obtain

$$T = \sum_{n=0}^{N} \sum_{k=0}^{n} \frac{(-1)^{n+k}}{(n-k)!} A_{2n+1,0}(r)\, t^{n-k} f_{k+1}(z,t) + T^{(N)}, \tag{3.2}$$

where

$$T^{(N)} = \frac{\varphi_{N+1}(z,t)}{(N+1)!} \int_0^\infty \lambda^{2N+3} \tilde{f}(\lambda) J_0(\lambda r)\, e^{-\lambda^2\xi} d\lambda. \tag{3.3}$$

It is seen from (3.3) that $T^{(N)} = O(\varphi_{N+1})$ $(t \to 0)$. Moreover, it can be shown that $\varphi_{n+1} = o(\varphi_n)$ $(t \to 0)$. Hence (3.2) yields an asymptotic expansion in the system of functions $\{\varphi_n\}$ as $t \to 0$.

Asymptotic expansions for the stresses are found analogously. The final answer has the form

$$T = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{(-1)^{n+k}}{(n-k)!} A_{2n+1,0}(r)\, t^{n-k} f_{k+1}(z,t), \tag{3.4}$$

$$\begin{aligned}
\sigma_{rr} = {} & -T + \operatorname{th}\{[B_{21}(r,z)z - 2(1-\mu)B_{11}(r,z)]/r - B_{30}(r,z)z + 2B_{20}(r,z)\} + \\
& + \sum_{n=0}^{\infty} \sum_{k=0}^{n} \sum_{m=0}^{n-k} \frac{(-1)^{m+n+k}}{(n-k-m)!} t^{n-k-m} \{(A_{2n+2,1}(r)/r - A_{2n+3,0}(r)) f_{k+m+2}(z,t) - \\
& - [(B_{2n+3,1}(r,z)z + B_{2n+2,1}(r,z)(hz-1) - 2hB_{2n+1,1}(r,z))/r - \\
& - B_{2n+4,0}(r,z)z - B_{2n+3,0}(r,z)(hz-1) + 2hB_{2n+2,0}(r,z) + \\
& + 2\mu(B_{2n+2,1}(r,z) + hB_{2n+1,1}(r,z))/r] f_{k+m+2}(0,t)\}, \\
\sigma_{\varphi\varphi} = {} & -T + \operatorname{th}\{[-B_{21}(r,z)z + 2(1-\mu)B_{11}(r,z)]/r + 2\mu B_{20}(r,z)\} + \\
& + \sum_{n=0}^{\infty} \sum_{k=0}^{n} \sum_{m=0}^{n-k} \frac{(-1)^{n+k+m}}{(n-k-m)!} t^{n-k-m} \{-A_{2n+2,1}(r)/r f_{k+m+2}(z,t) + \\
& + [2\mu([B_{2n+2,1}(r,z) + hB_{2n+1,1}(r,z)]/r - B_{2n+3,0}(r,z) - hB_{2n+2,0}(r,z)) + \\
& + (B_{2n+2,1}(r,z)[hz-1] + B_{2n+3,1}(r,z)z - 2hB_{2n+1,1}(r,z))/r] f_{k+m+2}(0,t)\},
\end{aligned}$$

$$\sigma_{zz} = \operatorname{th} B_{30}(r, z) + \sum_{n=0}^{\infty}\sum_{k=0}^{n}\sum_{m=0}^{n-k} \frac{(-1)^{k+m+n}}{(n-k-m)!} t^{n-k-m} \{A_{2n+3,0}(r) f_{k+m+2}(z, t) -$$
$$- [B_{2n+4,0}(r, z) z + B_{2n+3,0}(r, z)(1 + hz)] f_{k+m+2}(0, t)\},$$
$$\sigma_{rz} = [B_{31}(r, z) z - B_{21}(r, z)] \operatorname{th} + \sum_{n=0}^{\infty}\sum_{k=0}^{n}\sum_{m=0}^{n-k} \frac{(-1)^{k+m+n}}{(n-k-m)!} t^{n-k-m} \times$$
$$\times \{hA_{2n+2,1}(r) [\psi_{k+m+2}(z, t) - f_{k+m+2}(z, t)] - [B_{2n+3,1}(r, z) hz - B_{2n+2,1}(r, z) h + B_{2n+1,1}(r, z) z] f_{k+m+2}(0, t)\},$$

where

$$\psi_m(z, t) = L_t^{-1}\left[\frac{e^{-z\sqrt{s}}}{s^m}\right] \quad (m = 2, 3, \ldots), \tag{3.5}$$

and it can be shown on the basis of formulas in [4, 7] that

$$\psi_m(z, t) = \frac{(2t)^{m-1}}{(2m-2)!} e^{-\frac{z^2}{4t}} \left[\frac{d^{2m-2}}{dx^{2m-2}}\left(e^{\frac{x^2}{2}} \operatorname{erfc}\frac{x}{\sqrt{2}}\right)\right]_{x=\frac{z}{2\sqrt{t}}},$$
$$f_m(z, t) = \frac{1}{h^2} f_{m-1}(z, t) + \psi_m(z, t) + \frac{1}{h}\frac{\partial}{\partial z}\psi_m(z, t) \quad (m = 1, 2, \ldots),$$
$$f_0(z, t) = \frac{h}{\sqrt{\pi t}} e^{-\frac{z^2}{4t}} - he^{h^2t+hz}\operatorname{erfc}\left(\frac{z}{2\sqrt{t}} + h\sqrt{t}\right).$$

Going over from dimensionless stresses to dimensional stresses is realized by multiplying by $D = \alpha E\Theta_0(1 - \mu)^{-1}$, where E is Young's modulus, μ is the Poisson ratio, and α is the coefficient of linear temperature expansion.

The error that the asymptotic formulas (3.4) yield can be determined by estimating (3.3) and analogous integrals for the stress (see below).

4. Let us examine examples. In dimensional and dimensionless coordinates, take respectively

$$f(r) = \delta^3/\sqrt{(r^2 + \delta^2)^3},\ f(r) = 1/\sqrt{(r^2 + 1)^3}. \tag{4.1}$$

On the basis of relationships from [4] we find

$$\tilde{f}(\lambda) = e^{-\lambda},\ A_{ij} = F_{ij}(1),\ B_{ij} = F_{ij}(1 + z),$$
$$F_{ij}(x) = (-1)^i r^{-j}\frac{d^i}{dx^i}\left[\frac{\left(\sqrt{x^2 + r^2} - x\right)^j}{\sqrt{x^2 + r^2}}\right]. \tag{4.2}$$

Since $f_0 = \partial f_1/\partial t$, and $f_1(z, t)$ is the solution of the heat-conduction problem for a semibounded rod of initially zero temperature with heat transfer according to the Newton law from a medium of unit temperature at the end [8], then

$$f_0(z, t) \geqslant 0,\ 0 \leqslant f_1(z, t) \leqslant 1. \tag{4.3}$$

Using (4.3), we obtain an estimate of the error which (3.4) yields if N + 1 terms of the series (the N-approximation) are kept

$$T^{(N)} \leqslant \frac{(2N+3)!}{(N+1)!} t^{N+1} f_1(z, t),\ |\sigma_{rr}^{(N)}| \leqslant \frac{(2N+4)!}{(N+1)!}(2N + 5 + h)\left(\frac{7}{4} + \frac{1}{2e}\right) t^{N+2} f_1(0, t),$$
$$|\sigma_{\varphi\varphi}^{(N)}| \leqslant \frac{(2N+4)!}{(N+1)!}(2N + 5 + h)\left(2 + \frac{3}{4e}\right) t^{N+2} f_1(0, t),$$
$$|\sigma_{zz}^{(N)}| \leqslant \frac{(2N+4)!}{(N+1)!}\left(4N + 10 + \frac{h}{e}\right) t^{N+2} f_1(0, t),$$
$$|\sigma_{rz}^{(N)}| \leqslant \frac{(2N+4)!}{(N+1)!}[h + (2N + 5 + 2h) f_1(0, t)]\frac{1}{\sqrt{2}}.$$

For the Gauss distribution (1.1) the coefficients A_{ij} and B_{ij} are not expressed successfully in terms of tabulated functions; however, the asymptotic series (3.4) become convergent and the estimate of the approximations becomes considerably better

$$|T^{(N)}| \leqslant t^{N+1} f_1(z, t),\ |\sigma_{rr}^{(N)}| \leqslant (N+2)(h+1)\left(\frac{7}{4} + \frac{1}{2e}\right) t^{N+2} f_1(0, t),$$
$$|\sigma_{\varphi\varphi}^{(N)}| \leqslant (N+2)(h+1)\left(2 + \frac{3}{4e}\right) t^{N+2} f_1(0, t),\ |\sigma_{zz}^{(N)}| \leqslant (N+2)\left(2 + \frac{h}{e}\right) \times$$
$$\times\, t^{N+2} f_1(0, t),\ |\sigma_{rz}^{(N)}| \leqslant (N+2)\,[h + (2h+1) f_1(0, t)]\, t^{N+2} \frac{1}{\sqrt{2}}.$$

Moreover, the integrals in the expressions A_{ij} and B_{ij} converge quite rapidly because of the Gaussian exponential in the integrands. Hence, their computation, although more complex than by means of (4.2), is of no special difficulty.

Let us note that a computation of the stresses by means of (3.9) for the "domelike" Terezawa function [9] is just as simple as for (4.1) since even in this case the coefficients A_{ij} and B_{ij} are expressed in terms of elementary functions.

Calculations show that for $h < 50$ and $t < 0.01$ the zeroth approximations for (1.1) and (4.1) yield an error of several percent. The results for σ_{zz} and σ_{rz} turn out to be compressible only for small, and not even for all, values of z as is obtained in [3]. This is explained by the less successful selection of the system of functions σ_{rr} and $\sigma_{\varphi\varphi}$ used in our paper, which yields a smaller error of the approximations and permits a more exact clarification of the nature of the stress.

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